Math 1B Final Exam – Fall '08 Name______________________________

Write all responses on separate paper. Show your work for credit.

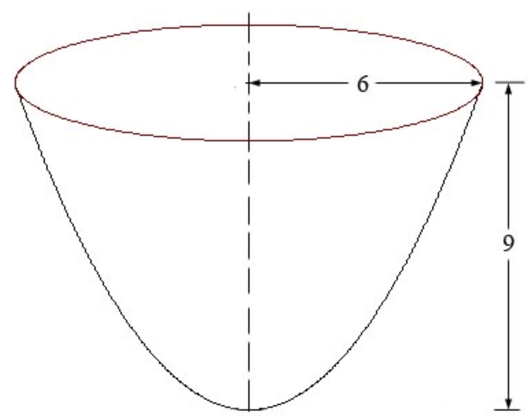
1. Evaluate the Riemann sum for $f(x) = x^2 + 2x$ with $0 \le x \le 2$
   a. Using four subintervals and right endpoints as sample points.
   b. Using four subintervals and midpoints as sample points.
   c. Using Simpson's rule.
   d. Draw a large, careful diagram to show what's going on in (a) and (b).
   e. Find the exact value of the integral and compare with the Riemann sums.

2. Evaluate the integral.

   a. $\int_0^{\pi/2} (1-\cos x)^9 \sin x \, dx$ b. $\int_0^\infty \frac{dx}{x^2+4}$ c. $\int_0^\infty e^{-2x}\cos(2x)\,dx$

3. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $0 \le x \le \frac{1}{2}$ and $2x \le y \le \sin \pi x$.

4. A tank full of water has the shape of a paraboloid with radius 6 meters at the top and a height of 9 meters, as shown at right.



   a. Find the work required to pump the water out of the top of the tank.

   b. After 6000 Joules of work has been done, what is the depth of the water remaining in the tank?

   c. What volume of paint is required to paint the outside of the paraboloid to a thickness of 4 mm?

5. Find the sum of the series.

   a. $\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^n}$ b. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n}(2n+1)!}$ c. $1 - \pi + \frac{\pi^2}{2} - \frac{\pi^3}{3!} + \frac{\pi^4}{4!} - \cdots$

6. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(x-1)^n$

7. Consider the integral $\int_0^1 \frac{x^5}{\sqrt{1+x^2}}\,dx$
   a. Use the substitution $u = 1 + x^2$ to find the antiderivative. You will need to express x^4 in terms of u.
   b. The value of the integral is $\frac{7\sqrt{2}-8}{15}$. Compare this value with the value obtained by approximating the integrand by the first four terms of its Maclaurin series and then integrating.

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1. Evaluate the Riemann sum for $f(x) = x^2 + 2x$ with $0 \le x \le 2$

   a. Using four subintervals and right endpoints as sample points.

   SOLN: $$\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x = \sum_{k=1}^{4} f\left(\frac{k}{2}\right)\frac{1}{2} = \frac{1}{2}\sum_{k=1}^{4}\left(\frac{k}{2}\right)^2 + k = \frac{1}{8}\sum_{k=1}^{4}k^2 + \frac{1}{2}\sum_{k=1}^{4}k$$
   $$= \frac{1}{8}\frac{4(4+1)(8+1)}{6} + \frac{1}{2}\frac{4(4+1)}{2} = \frac{15}{4} + 5 = \frac{35}{4}$$

   b. Using four subintervals and midpoints as sample points.

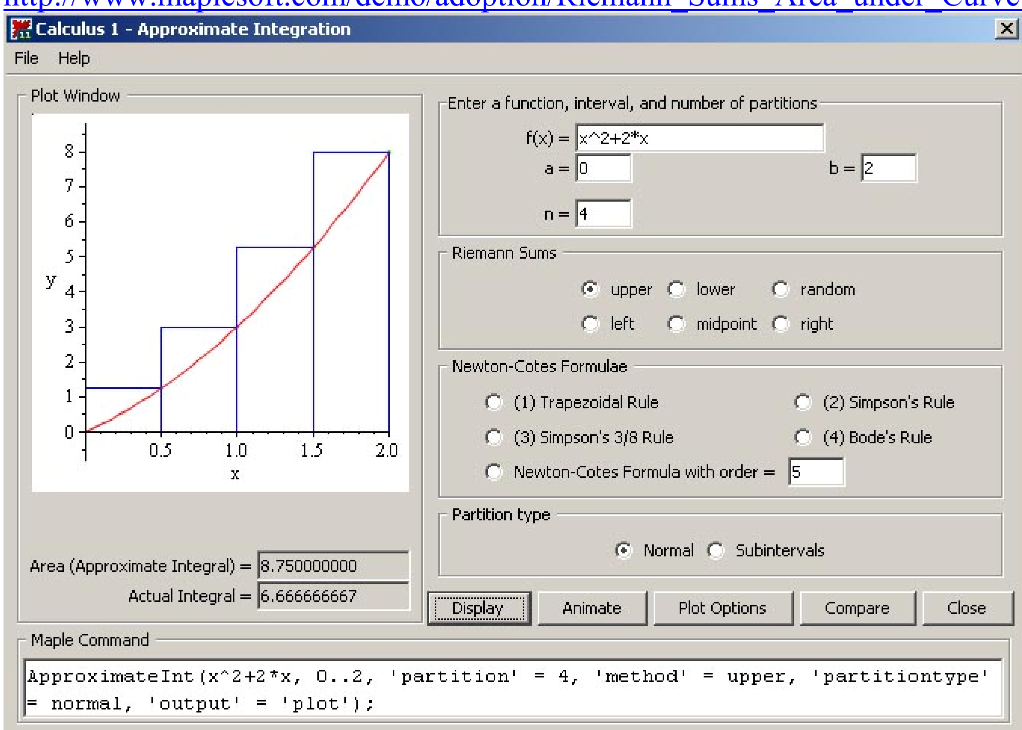
   SOLN:
   $$\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x = \sum_{k=1}^{4} f\left(\frac{k}{2}-\frac{1}{4}\right)\frac{1}{2} = \frac{1}{2}\sum_{k=1}^{4}\left(\frac{k}{2}-\frac{1}{4}\right)^2 + \left(k-\frac{1}{2}\right) = \frac{1}{32}\sum_{k=1}^{4}(2k-1)^2 + \left(\frac{1}{2}\sum_{k=1}^{4}k\right) - 1$$
   $$= \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right)\right) = \frac{1}{2}\left(\frac{9}{16} + \frac{33}{16} + \frac{65}{16} + \frac{105}{16}\right) = \frac{53}{8}$$

   c. Using Simpson's rule.

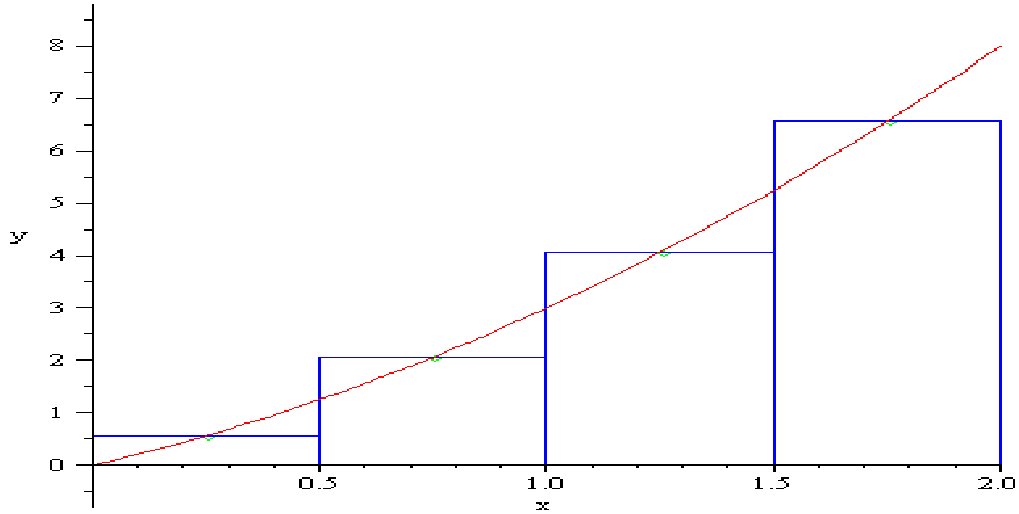
   SOLN:
   $$\frac{1}{12}\left(f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + 2f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right)$$
   $$= \frac{1}{12}\left(0 + \frac{9}{4} + \frac{5}{2} + \frac{33}{4} + 6 + \frac{65}{4} + \frac{21}{2} + \frac{105}{4} + 8\right) = \frac{20}{3}$$

   d. Draw a large, careful diagram to show what's going on in (a) and (b).

   SOLN: Here's the result of using Maple 11's Riemann Sum Tutor from the Context Menu as described in the video at
   http://www.maplesoft.com/demo/adoption/Riemann_Sums_Area_under_Curve.aspx



You can also choose midpoint radio button with these settings and get the diagram below:



e. Find the exact value of the integral and compare with the Riemann sums.

SOLN: $\int_0^2 x^2+2x\,dx=\frac{x^3}{3}+x^2\Big|_0^2=\frac{8}{3}+4=\frac{20}{3}$ matches the Simpson value exactly (as is true for all quadratic integrands) . The right endpoint sum is an over-estimate because the function is increasing and the midpoint sum is an under-estimate because the function is concave up.

2. Evaluate the integral.

a. $\int_0^{\pi/2}(1-\cos x)^9\sin x\,dx$

SOLN: Substitute $u = 1 - \cos x$ and $du = \sin x\,dx$ to get

$$\int_0^{\pi/2}(1-\cos x)^9\sin x\,dx=\int_0^1 u^9\,du=\frac{u^{10}}{10}\Big|_0^1=\frac{1}{10}$$

b. $\int_0^\infty \frac{dx}{x^2+4}$

SOLN: Substitute $x = 2u$ and $dx = 2du$ to get $\int_0^\infty \frac{dx}{x^2+4}=\int_0^\infty\frac{2du}{4u^2+4}=\frac{1}{2}\lim_{b\to\infty}\arctan u\Big|_0^b=\frac{\pi}{4}$

c. $\int_0^\infty e^{-2x}\cos(2x)\,dx$

SOLN. Let I = the value of the integral. By parts,

$$I=\frac{1}{2}\sin(2x)e^{-2x}\Big|_0^\infty+\int_0^\infty e^{-2x}\sin(2x)\,dx=0-0-\frac{1}{2}\cos(2x)e^{-2x}\Big|_0^\infty-I\text{, whence } 2I=\frac{1}{2}\Leftrightarrow I=\frac{1}{4}$$

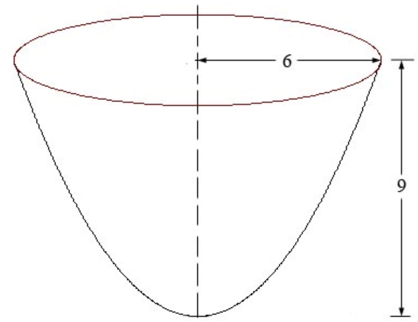
3. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $0\le x\le\frac{1}{2}$ and $2x\le y\le\sin\pi x$.

SOLN:

$$2\pi\int_0^{1/2} rh\,dx=2\pi\int_0^{1/2}x(\sin\pi x-2x)\,dx=-2x\cos\pi x\Big|_0^{1/2}+2\int_0^{1/2}(\cos\pi x)\,dx-\frac{4}{3}\pi x^3\Big|_0^{1/2}$$

$$=\frac{2}{\pi}\sin\pi x\Big|_0^{1/2}-\frac{\pi}{6}=\frac{2}{\pi}-\frac{\pi}{6}\approx 0.1130$$

4. A tank full of water has the shape of a paraboloid with radius 6 meters at the top and a height of 9 meters, as shown at right.



a. Find the work required to pump the water out of the top of the tank. SOLN:

$$W = \int dW = \int y dF = 9810\pi \int_0^9 (9-y)x^2 dy$$

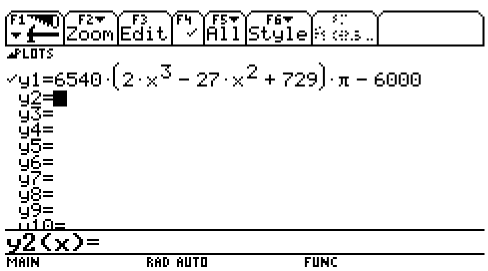
$$= 39240\pi \int_0^9 (9-y)y dy = 39240\pi \left(\frac{9y^2}{2} - \frac{y^3}{3} \right) \Bigg|_0^9$$

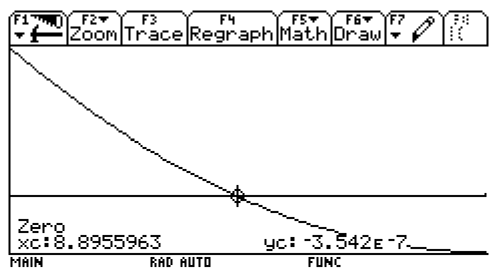
$$= 39240\pi \left(\frac{729}{2} - \frac{729}{3} \right) = 39240\pi \left(\frac{729}{6} \right) = 4767660\pi \text{ Joules}$$

b. After 6000 Joules of work has been done, what is the depth of the water remaining in the tank?

SOLN: $39240\pi \int_d^9 (9-y) y dy = 39240\pi \left(\frac{9y^2}{2} - \frac{y^3}{3} \right) \Bigg|_d^9 = 39240\pi \left(\frac{729}{6} + \frac{d^3}{3} - \frac{9d^2}{2} \right) = 6000$

This is equivalent to $6540\pi\left(2d^3 - 27d^2 + 729\right) = 6000$. It is appropriate to use a calculator to approximate the solution. In the screen shoots from the TI92 shown below, you can see $y1$ is set to be the difference between the right and left sides of the equation and that the zero is found on the graph at approximately d = 8.8955963 meters.





c. What volume of paint is required to paint the outside of the paraboloid to a thickness of 4 mm?
SOLN: The curve is $y = x^2/4$ The surface area is

$$2\pi \int r ds = 2\pi \int_0^6 x\sqrt{1 + (dy/dx)^2} dx = \pi \int_0^6 x\sqrt{4 + x^2} dx = \frac{\pi}{2} \int_4^{40} \sqrt{u}\, du$$

$$= \frac{\pi}{3}\left(40^{3/2} - 4^{3/2}\right) = \frac{\pi}{3}\left(80\sqrt{10} - 8\right) \approx 256.5 \text{ m}^2$$

Multiplying by 4 mm yields approximately 1.026 cubic meters of paint.

5. Find the sum of the series.

a. $\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^n}$

SOLN: $\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^n} = 3\sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{3}{1 - 9/10} = 30$

b. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1}}{3^{2n}(2n+1)!}$

SOLN: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1}}{3^{2n}(2n+1)!} = 3\sum_{n=0}^{\infty}\frac{(-1)^n (\pi/3)^{2n+1}}{(2n+1)!} = 3\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$

c. $1-\pi+\frac{\pi^2}{2}-\frac{\pi^3}{3!}+\frac{\pi^4}{4!}-\cdots$

SOLN: $\displaystyle 1-\pi+\frac{\pi^2}{2}-\frac{\pi^3}{3!}+\frac{\pi^4}{4!}-\cdots=\sum_{n=0}^{\infty}\frac{(-\pi)^n}{n!}=e^{-\pi}=\frac{1}{e^{\pi}}$

6. Find the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}(x-1)^n$

SOLN: Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{(2n+2)!}{((n+1)!)^2}\frac{(x-1)^{n+1}}{(x-1)^n}\frac{(n!)^2}{(2n)!}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}|x-1| = 4|x-1|<1 \Leftrightarrow x\in\left(\frac{3}{4},\frac{5}{4}\right)$$

At $x = ¾$, the series is alternating, but $(2n)! > (n!)^2$, so the series is divergent by the nth term test for both endpoints, therefore the interval of convergence is just (0.75, 1.25).

7. Consider the integral $\displaystyle\int_0^1 \frac{x^5}{\sqrt{1+x^2}}dx$

a. Use the substitution $u = 1 + x^2$ to find the antiderivative.
You will need to express x^4 in terms of u.

SOLN:
$$\int\frac{x^5}{\sqrt{1+x^2}}dx = \frac{1}{2}\int\frac{(u-1)^2}{\sqrt{u}}du = \frac{1}{2}\int u^{3/2}-2u^{1/2}+u^{-1/2}du$$
$$=\left(\frac{u^2}{5}-\frac{2u}{3}+1\right)u^{1/2} = \left(\frac{(1+x^2)^2}{5}-\frac{2(1+x^2)}{3}+1\right)\sqrt{1+x^2}$$

b. The value of the integral is $\frac{7\sqrt{2}-8}{15}$. Compare this value with the value obtained by approximating the integrand by the first four terms of its Maclaurin series and then integrating.

SOLN:
$$\int_0^1\frac{x^5}{\sqrt{1+x^2}}dx \approx \int_0^1 x^5\sum_{k=0}^{3}\binom{-1/2}{k}x^{2k}dx = \sum_{k=0}^{3}\binom{-1/2}{k}\int_0^1 x^{2k+5}dx = \sum_{k=0}^{3}\binom{-1/2}{k}\frac{1}{2k+6}$$
$$=\sum_{k=0}^{3}\binom{-1/2}{k}\frac{1}{2k+6} = \frac{1}{6}-\frac{1}{16}+\frac{3}{80}-\frac{5}{192}=\frac{37}{320}=0.115625$$

The last term in the alternating series has magnitude of about 0.026 > error in approx ~ 0.011